

A NOTE ON THE HOMOTOPY TYPE OF THE ALEXANDER DUAL

ELÍAS GABRIEL MINIAN AND JORGE TOMÁS RODRÍGUEZ

ABSTRACT. We investigate the homotopy type of the Alexander dual of a simplicial complex. In general the homotopy type of K does not determine the homotopy type of its dual K^* . Moreover, one can construct for each finitely presented group G , a simply connected simplicial complex K such that $\pi_1(K^*) = G$. We study sufficient conditions on K for K^* to have the homotopy type of a sphere. We also extend the simplicial Alexander duality to the context of reduced lattices.

1. INTRODUCTION

Let A be a compact and locally contractible proper subspace of S^n . The classical Alexander duality theorem asserts that the reduced homology groups $H_i(S^n - A)$ are isomorphic to the reduced cohomology groups $H^{n-i-1}(A)$ (see for example [6, Thm. 3.44]). The combinatorial (or simplicial) Alexander duality is a special case of the classical duality: if K is a finite simplicial complex and K^* is the Alexander dual with respect to a ground set $V \supseteq K^0$, then for any i

$$H_i(K) \cong H^{n-i-3}(K^*).$$

Here K^0 denotes the set of vertices (i.e. 0-simplices) of K and n is the size of V . A very nice and simple proof of the combinatorial Alexander duality can be found in [4]. An alternative proof of this combinatorial duality can be found in [3].

In these notes we relate the homotopy type of K with that of K^* . We show first that, even though the homology of K determines the homology of K^* (and vice versa), the homotopy type of K does not determine the homotopy type of K^* . Moreover, for any finitely presented group G , one can find a simply connected complex K such that its Alexander dual, with respect to some ground set V , has fundamental group isomorphic to G . In the same direction, we exhibit an example of a complex with the homotopy type of a sphere whose dual is not homotopy equivalent to a sphere. However we prove that if K simplicially collapses to the boundary of a simplex, then K^* is homotopy equivalent to a sphere. To prove this result we use the nerve of the dual. We also use the nerve to find an easy-to-check sufficient condition for a complex to simplicially collapse to the boundary of a simplex.

In the last section of these notes we extend the duality to the context of reduced lattices. A reduced lattice is a finite poset with the property that any subset which is bounded below has an infimum. Any finite simplicial complex can be seen as a reduced lattice by means of its face poset. We define the Alexander dual for reduced lattices and show that

2000 *Mathematics Subject Classification.* 55U05, 55P15, 57Q05, 57Q10, 55M05, 06A06.

Key words and phrases. Alexander duality, simplicial complexes, lattices, homotopy types, finite topological spaces.

the duality theorem remains valid in this context. When the poset is the face poset of a simplicial complex, the construction coincides with the simplicial one.

2. PRELIMINARIES

Let K be finite simplicial complex and let V be a set which contains the set K^0 of 0-simplices of K . The Alexander dual of K (with respect to the fixed set V) is the simplicial complex

$$K^* = \{\sigma \subset V, \sigma^c \notin K\}.$$

Here $\sigma^c = V \setminus \sigma$, the complement of σ in V . It is clear that $K^{**} = K$. Note that the set V is implicit in the definition of the dual.

The simplicial Alexander dual K^* allows us to investigate the homology of K but in general the homotopy type of K^* does not determine the homotopy type of K . Moreover, the fundamental group of K^* does not provide information about the fundamental group of K . In fact, one can prove the following.

Proposition 2.1. *For any given finitely presented group G , there exists a connected compact simplicial complex K such that $\pi_1(K) = G$ and such that its Alexander dual K^* with respect to any $V \supseteq K^0$ is simply connected.*

Proof. Since G is finitely presented, there exists a connected 2-dimensional finite simplicial complex K such that $\pi_1(K) = G$. We can suppose without loss of generality that K has more than six vertices. The dual of K , with respect to any $V \supseteq K^0$ contains the whole 2-skeleton of the simplex spanned by V , since the complement of any subset of three elements of V is not a simplex in K , by a cardinality argument. It follows that K^* is simply connected. \square

Corollary 2.2. *For any finitely presented group G there is a simply connected complex whose dual, with respect to some V , has fundamental group isomorphic to G .*

In the same direction, the following example shows two homotopy equivalent simplicial complexes K, L such that $K^0 = L^0 = V$ and such that their duals K^*, L^* (with respect to V) are not homotopy equivalent.

Example 2.3. Let M be a triangulation of the Poincaré homology 3-sphere and let S be the boundary of a 4-simplex whose vertices are contained in the set $V = M^0$. Similarly as in the proof of Proposition 2.1, since any triangulation M of the homology 3-sphere has more than 7 vertices and M and S are 3-dimensional, their duals $K = M^*$ and $L = S^*$ (with respect to V) are simply connected. Since K and L have the homology of a sphere S^n , it follows that they are in fact homotopy equivalent. Moreover, $K^0 = L^0 = V$ and their duals are respectively M and S , which are not homotopy equivalent.

In particular, the last example shows that the dual of a complex which is homotopy equivalent to a sphere need not be homotopy equivalent to a sphere. The next lemma shows that, when we restrict ourselves to simplicial collapses, the duals preserve the homotopy type. We refer the reader to [5] for the basic notions on simplicial collapses and expansions and simple homotopy types. As usual, we will denote an elementary simplicial collapse by $K \searrow L$ and, in general, $K \searrow L$ will denote a simplicial collapse.

Lemma 2.4. *Let L be a subcomplex of K and let V be a set containing K^0 . Then $K \searrow L$ if and only if $K^* \nearrow L^*$. Consequently, if $K \searrow L$, then $K^* \nearrow L^*$.*

Proof. Note that if $L = K \setminus \{\tau, \sigma\}$ with τ a free face of σ , then $K^* = L^* \setminus \{\sigma^c, \tau^c\}$ with σ^c a free face of τ^c . \square

Recall that the nerve $N(K)$ of a simplicial complex K is the complex whose vertices are the maximal simplices (=facets) of K and the simplices are the subsets of facets with non-empty intersection. It is well-known that $N(K)$ is homotopy equivalent to K .

Lemma 2.5. *Let $\dot{\tau}$ be the boundary of a simplex and let V be a set such that $\tau^0 \subsetneq V$. Then $(\dot{\tau})^*$ is homotopy equivalent to the sphere S^{n-1} , where $n = \#V - \#\tau^0$.*

Proof. If $n = 1$, $V = \tau^0 \cup \{v\}$ and $(\dot{\tau})^*$ is the disjoint union of the simplex τ and the vertex v . Then $(\dot{\tau})^*$ is homotopy equivalent to S^0 .

In general, if $V = \tau^0 \cup \{v_1, \dots, v_n\}$, $(\dot{\tau})^*$ has $n + 1$ maximal simplices, namely the simplices η_i with vertex sets $\tau^0 \cup \{v_1, \dots, \hat{v}_i, \dots, v_n\}$, for $i = 1, \dots, n$, and η_{n+1} whose vertex set is $\{v_1, \dots, v_n\}$. The intersection of all these simplices is empty but any other intersection is non-empty. Then the nerve of $(\dot{\tau})^*$ is the boundary of the n -simplex and therefore $(\dot{\tau})^*$ is homotopy equivalent to S^{n-1} . \square

Corollary 2.6. *If K collapses to the boundary of a simplex, then K^* is homotopy equivalent to a sphere.*

We can use the nerve of the complex to find an easy-to-check sufficient condition for a complex to collapse to the boundary of a simplex. Note that in many cases, the nerve of a complex K is much smaller than K . Moreover, in [2] it is proved that any complex K *strong collapses* to the square-nerve $N^2(K) = N(N(K))$. In particular, $K \searrow N^2(K)$. The strong collapses are easier to handle than the usual collapses. The concrete definition is the following.

Definition 2.7. Let K be a complex and let $v \in K$ be a vertex. We denote by $K \setminus v$ the full subcomplex of K spanned by the vertices different from v (the *deletion* of the vertex v). We say that there is an *elementary strong collapse* from K to $K \setminus v$ if the link of the vertex $lk(v, K)$ is a simplicial cone $v'L$, for some v' . In this case we say that v is *dominated* (by v') and we denote $K \searrow_e K \setminus v$. There is a *strong collapse* from a complex K to a subcomplex L if there exists a sequence of elementary strong collapses that starts in K and ends in L . In this case we write $K \searrow L$.

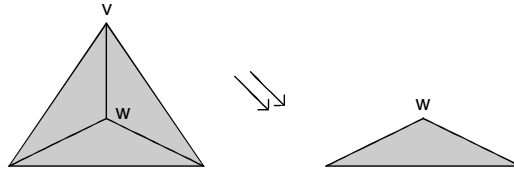


FIGURE 1. An elementary strong collapse.

It is easy to see that $K \searrow_e L$ implies $K \searrow L$. We refer the reader to [2] for a comprehensive exposition on strong collapsibility and its relationship with simplicial collapsibility. The following lemma shows that this kind of collapses behaves well with respect to the nerve construction.

Lemma 2.8. *If $K \searrow_e L$, then $N(K) \searrow_e N(L)$.*

Proof. We may suppose that $K \searrow^e L$, i.e. $L = K \setminus \{v\}$ with the vertex v dominated by w . Consider the simplicial map $f : N(L) \rightarrow N(K)$ defined in the vertices of $N(L)$ by

$$f(\sigma) = \begin{cases} \sigma & \text{if } \sigma \in N(K) \\ v\sigma & \text{if } \sigma \notin N(K) \end{cases}$$

It is easy to see that $\bigcap \sigma_i \neq \phi$ if and only if $\bigcap f(\sigma_i) \neq \phi$. Therefore we only need to prove that $N(K) \searrow f(N(L))$. By [2, Lemma 3.3], it suffices to check that every vertex $\gamma \in N(K) \setminus f(N(L))$ is dominated by a vertex of $f(N(L))$.

Let γ be a vertex in $N(K) \setminus f(N(L))$. Since $\gamma \notin f(N(L))$, then $\gamma = v\gamma'$ with γ' not maximal in L . Therefore there exists $\tau \in L$ a maximal simplex with $\gamma' \subsetneq \tau$. We will show that γ is dominated by τ in $N(K)$.

Let $\{\sigma_0, \dots, \sigma_l\} \in lk(\gamma, N(K))$ (i.e. $\bigcap \sigma_i \cap \gamma \neq \phi$). We need to prove that $\bigcap \sigma_i \cap \tau \neq \phi$. If $v \in \bigcap \sigma_i \cap \gamma$, then $v \in \sigma_i$. Since w dominates v and σ_i is maximal in K , we conclude that $w \in \sigma_i$ and therefore $w \in \bigcap \sigma_i \cap \tau$. If $v \notin \bigcap \sigma_i \cap \gamma$, then $\bigcap \sigma_i \cap \gamma \subseteq \gamma'$. Since $\gamma' \subsetneq \tau$, it follows that $\bigcap \sigma_i \cap \tau \neq \phi$ □

Note that in general the previous lemma is not true for simplicial collapses.

Corollary 2.9. *Let K be a simplicial complex such that $N(K) \searrow \dot{\sigma}$, where $\dot{\sigma}$ is the boundary of a simplex. Then $K \searrow \dot{\sigma}$, and therefore K^* is homotopy equivalent to a sphere.*

Proof. By Lemma 2.8, $N(N(K)) \searrow N(\dot{\sigma}) = \dot{\sigma}$ and by [2, Proposition 3.4], $K \searrow N^2(K)$. It follows that $K \searrow \dot{\sigma}$ and, in particular, $K \searrow \dot{\sigma}$. □

3. THE DUALITY IN TERMS OF REDUCED LATTICES

Definition 3.1. A finite poset X is called a *reduced lattice* if every lower bounded set of X has an infimum.

Equivalently, a poset is a reduced lattice if and only if it is obtained from a finite lattice by deleting the maximum and the minimum. Note that if X is a reduced lattice, every upper bounded set has a supremum. For example, the face poset $\mathcal{X}(K)$ of any finite simplicial complex K is a reduced lattice.

Definition 3.2. Given a reduced lattice X , we denote by $m(X)$ the set of its minimal elements and by $T(X)$ the simplicial complex whose vertex set is $m(X)$ and whose simplices are the subsets of $m(X)$ which are bounded above.

Note that this construction is related to the \mathcal{L} -complex defined in [1, Section 9.2]. In fact, $T(X) = \mathcal{L}(X^{op})$, the \mathcal{L} -complex of the opposite of X .

Remark 3.3. It is clear that $T(\mathcal{X}(K)) = K$ for any finite simplicial complex K . Moreover, by [1, Section 9.2], for any reduced lattice X , the complex $T(X)$ is homotopy equivalent to the standard order complex $\mathcal{K}(X)$ whose simplices are the non-empty chains of X .

Definition 3.4. Given a reduced lattice X and a set V such that $m(X) \subseteq V$, we define its Alexander dual as the reduced lattice $\mathcal{X}(T(X)^*)$. Here $T(X)^*$ denotes the Alexander dual of the simplicial complex $T(X)$ with respect to the ground set V .

By Remark 3.3, the simplicial Alexander duality immediately extends to this context as follows.

Proposition 3.5. *Given a reduced lattice X and a set V such that $m(X) \subseteq V$, then for any i*

$$H_i(X) \cong H^{n-i-3}(X^*),$$

where $n = \#V$.

The (co)homology of a poset X is the (co)homology of its associated order complex $\mathcal{K}(X)$. It is known that a finite poset is essentially a finite topological space and its homology groups coincide with the homology groups of the associated order complex (see [1, 2]). Therefore this result also can be used to investigate the topology of finite spaces.

Remark 3.6. Since $K = T(\mathcal{K}(K))$, this version of the duality extends the simplicial version. Note also that in general $X^{**} \neq X$, unless $X = \mathcal{K}(K)$ for some simplicial complex K . In fact, $X^{**} = \mathcal{K}(T(X))$.

Example 3.7. Figure 2 shows a reduced lattice X , which is not the face poset of a complex, and its dual X^* .

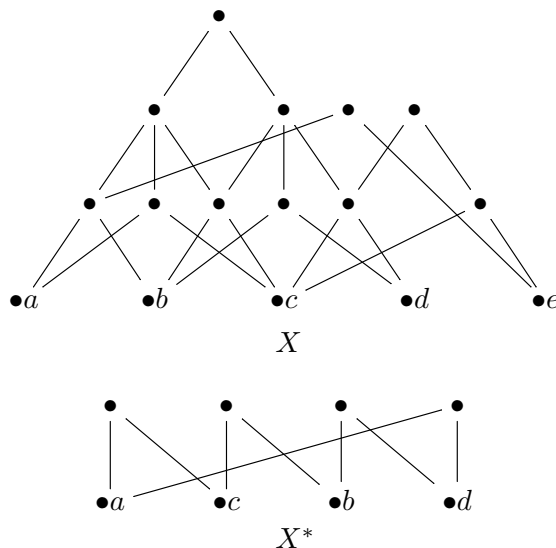


FIGURE 2. A reduced lattice and its dual.

REFERENCES

- [1] J.A. Barmak. *Algebraic topology of finite topological spaces and applications*. Lecture Notes in Mathematics Vol. 2032. Springer (2011).
- [2] J.A. Barmak and E.G. Minian. *Strong homotopy types, nerves and collapses*. Discrete Comput. Geom. 47 (2012), no. 2, 301-328.
- [3] M. Barr. *A duality on simplicial complexes*. Georgian Math. J. 9 (2002), no. 4, 601-605.
- [4] A. Björner and M. Tancer. *Combinatorial Alexander duality. A short and elementary proof*. Discrete Comput. Geom. 42 (2009), no. 4, 586-593.
- [5] M.M. Cohen. *A Course in Simple Homotopy Theory*. Springer-Verlag New York, Heidelberg, Berlin (1970).

[6] A. Hatcher. *Algebraic topology*. Cambridge University Press (2002).

DEPARTAMENTO DE MATEMÁTICA–IMAS, FCEYN, UNIVERSIDAD DE BUENOS AIRES, BUENOS AIRES,
ARGENTINA.

E-mail address: gminian@dm.uba.ar

E-mail address: jtrodrig@dm.uba.ar